

ON ARITHMETIC SUMS INVOLVING DIVISOR FUNCTIONS IN TWO VARIABLES

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ABSTRACT. We prove the analogue of an identity of Huard, Ou, Spearman and Williams and apply it to evaluate a variety of sums involving divisor functions in two variables. It turns out that these sums count representations of positive integers involving radicals.

1. INTRODUCTION

Throughout, let \mathbb{N} , $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{Z} , \mathbb{Q} , and \mathbb{C} be the sets of positive integers, nonnegative integers, integers, rational numbers, and complex numbers respectively. Let $\sigma_k(n)$ be the sum of k th powers of the divisors of n with the assumption that $\sigma_k(n) = 0$ if $n \notin \mathbb{N}$. Convolution sums of the form

$$\sum_{m=1}^{n-1} \sigma_r(m) \sigma_s(n-m), \quad (r, s \in \mathbb{N})$$

have been investigated by many mathematicians including in chronicle order Glaisher [2, 3, 4], Ramanujan [10], MacMahon [8], Lahiri [6, 7], and Melfi [9]. A variety of such convolution sums have been evaluated using advanced techniques such as the theory of modular forms and the theory of q -series. In reference [5], Huard, Ou, Spearman, and Williams among other things gave elementary proofs for many of these convolution sums along with many new identities. See also Williams [12, Chapter 13]. The authors' main argument is the following theorem.

Theorem 1. *Let $2 \leq n \in \mathbb{N}$ and let*

$$\mathcal{B}(n) = \{(a, b, x, y) \in \mathbb{N}^4 : ax + by = n\}.$$

Let $f : \mathbb{Z}^4 \rightarrow \mathbb{C}$ be such that

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)$$

for all $a, b, x, y \in \mathbb{Z}$. Then

$$\begin{aligned} & \sum_{(a,b,x,y) \in \mathcal{B}(n)} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y) \\ & - f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y)) \\ & = \sum_{d \in \mathbb{N}, d|n} \sum_{x \in \mathbb{N}, x < d} (f(0, n/d, x, d) + f(n/d, 0, d, x) + f(n/d, n/d, d-x, -x)) \end{aligned}$$

Date: November 26, 2012.

1991 Mathematics Subject Classification. 11A25, 11B75, 11B50.

Key words and phrases. Arithmetic sums, convolution sums, divisor functions, integer representations.

$$-f(x, x-d, n/d, n/d) - f(x, d, 0, n/d) - f(d, x, n/d, 0)).$$

For instance, by application of Theorem 1 to the function $f(a, b, x, y) = xy$ the authors obtained the Besge's identity

$$\sum_{m=1}^{n-1} \sigma(m)\sigma(n-m) = \frac{1}{12}(5\sigma_3(n) + (1-6n)\sigma(n))$$

and by application of the same theorem to the function $f(a, b, x, y) = x^3y + xy^3$ the authors reproduced the Glaisher's formula

$$\sum_{m=1}^{n-1} \sigma(m)\sigma_3(n-m) = \frac{1}{240}(21\sigma_5(n) + (10-30n)\sigma_3(n) - \sigma(n)).$$

They also used Theorem 1 to deduce formulas for the sums

$$\sum_{m=1}^{n-1} \sigma_r(m)\sigma_s(n-m)$$

for odd r and s such that $r+s \in \{2, 4, 6, 8, 12\}$. However, as was pointed out by the authors formulas for such sums for odd r and s such that $r+s = 10$ remain out of reach of their methods. These had been evaluated earlier by Lahiri [7] using Ramanujan's tau function $\tau(n)$. Quite recently, Royer [11] used quasimodular forms to reproduce many convolution sums involving the divisor functions.

It is natural to ask what would happen if in Theorem 1 for instance instead of the set $\mathcal{B}(n)$ one takes the sum over the set

$$\mathcal{B}'(n) = \{(a, b, x, y) \in \mathbb{N}^4 : ax + by = n, \text{ and } \gcd(a, b) = \gcd(x, y) = 1\}.$$

The main theorem of our paper deals with such an analogue of Theorem 1.

Main Theorem. *Let $2 \leq n \in \mathbb{N}$ and let $f : \mathbb{Z}^4 \rightarrow \mathbb{C}$ be such that*

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)$$

for all $a, b, x, y \in \mathbb{Z}$. Then

$$\begin{aligned} (1) \quad & \sum_{(a,b,x,y) \in \mathcal{B}'(n)} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y) \\ & - f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y)) \\ & = \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (f(1, 0, n, t) - f(n, t, 1, 0) + f(0, 1, t, n) \\ & - f(t, n, 0, 1) + f(1, 1, n-t, -t) - f(n-t, -t, 1, 1)). \end{aligned}$$

Our work suggests the following definition of the sum of divisors function in two variables.

Definition 1. For $m, n \in \mathbb{N}$ and $r, s \in \mathbb{N}_0$ let

$$\sigma'_{r,s}(m, n) = \sigma'_{s,r}(n, m) = \sum_{\substack{(d,e) \in \mathbb{N}^2 \\ d|m, e|n \\ (d,e) = (\frac{m}{d}, \frac{n}{e}) = 1}} d^r e^s.$$

It is easily seen that

$$(2) \quad \sum_{m=1}^{n-1} \sigma'_{r,s}(m, n-m) = \sum_{m=1}^{n-1} \sigma'_{s,r}(m, n-m)$$

$$= \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^r y^s = \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^s y^r = \sum_{(a,b,x,y) \in \mathcal{B}'(n)} a^r b^s = \sum_{(a,b,x,y) \in \mathcal{B}'(n)} a^s b^r.$$

We shall use Main Theorem and the relation (2) to evaluate the sums

$$\sum_{m=1}^{n-1} \sigma'_{r,s}(m, n-m)$$

for $r \equiv s \equiv 1 \pmod{2}$ such that $r+s \in \{2, 4, 6, 8, 12\}$. Whereas the sums $\sum_{m=1}^{n-1} \sigma_r(m) \sigma_s(n-m)$ for these values of r and s are in terms of the divisor functions $\sigma_k(n)$, their analogues $\sum_{m=1}^{n-1} \sigma'_{r,s}(m, n-m)$ are in terms of the function $\psi_s(n)$ defined by

$$\psi_s(n) = \sum_{d|n} \mu(d) d^s \quad \text{for } n \in \mathbb{N} \text{ and } s \in \mathbb{Z} \setminus \{0\},$$

where $\mu(n)$ denotes the Möbius mu function. To be fair to Huard, Ou, Spearman, and Williams we note that our proofs are essentially the same as theirs for the corresponding results in reference [5]. As a new development, we will show that our identities actually count representations of positive integers.

Remark. While formulas are known for the sums $\sum_{m=1}^{n-1} \sigma_r(m) \sigma_s(n-m)$ for odd r and s such that $r+s=10$, to the authors' knowledge no formulas are known for the corresponding sums $\sum_{m=1}^{n-1} \sigma'_{r,s}(m, n-m)$ for the same r and s .

Throughout n will denote a positive integer which is greater than 1 and p will denote a prime number. We shall sometimes write (m, n) for $\gcd(m, n)$. Next $\phi(n)$ will denote the Euler totient function. For our current purposes we record the following well-known properties of these two functions. If $n > 1$ and $s \in \mathbb{Z} \setminus \{0\}$, then

$$(3) \quad \psi_s(n) = \prod_{p|n} (1 - p^s), \quad \phi(n) = n \psi_{-1}(n), \quad \sum_{d|n} \phi(d) = n, \quad \sum_{d|n} \mu(d) = 0.$$

Note that

$$(4) \quad \sum_{\substack{(x,y) \in \mathbb{N}^2 \\ x+y=n \\ \gcd(x,y)=1}} 1 = \phi(n), \quad \text{for } n > 1 \quad \text{and} \quad \sum_{\substack{(x,y) \in \mathbb{N}_0 \times \mathbb{N} \\ x+y=n \\ \gcd(x,y)=1}} 1 = \phi(n) \quad \text{for } n \geq 1.$$

Further, it is easy to verify the following formula which is crucial in this paper. If $k, n \in \mathbb{N}$ such that $n > 1$, then

$$(5) \quad \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^k = \sum_{d|n} \mu(d) d^k \sum_{j=1}^{\frac{n}{d}} j^k = \sum_{d|n} \mu(d) d^k \sum_{j=1}^{\frac{n}{d}-1} j^k$$

$$= \sum_{d|n} \mu(d) \frac{d^k}{k+1} \sum_{j=0}^k \binom{k+1}{j} B_j \left(\frac{n}{d} \right)^{k+1-j},$$

where B_j denotes the j th Bernoulli number for which $B_1 = -1/2$ and $B_{2j+1} = 0$ for $j \in \mathbb{N}$ and the first few terms for even j are

$$B_0 = 1, B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30, B_{10} = \frac{5}{66}, B_{12} = \frac{-691}{2730}.$$

We now state the applications of Main Theorem which we intend to prove in this work.

Theorem 2. [1, Theorem 1 (a)] *We have*

$$\sum_{m=1}^{n-1} \sigma'_{1,1}(m, n-m) = \frac{5n^3 - 6n}{12} \psi_{-1}(n) + \frac{n}{12} \psi_1(n).$$

Theorem 3. *We have*

$$\sum_{m=1}^{n-1} \sigma'_{1,3}(m, n-m) = \frac{7n^5 - 10n}{10} \psi_{-1}(n) + \frac{n^3}{3} \psi_1(n) - \frac{n}{30} \psi_3(n).$$

Theorem 4. *We have*

$$(a) \sum_{m=1}^{n-1} \sigma'_{1,5}(m, n-m) = \frac{540n^7 - 1134n}{13608} \psi_{-1}(n) + \frac{n^5}{24} \psi_1(n) + \frac{9n}{4536} \psi_5(n),$$

$$(b) \sum_{m=1}^{n-1} \sigma'_{3,3}(m, n-m) = \frac{n^7}{120} \psi_{-1}(n) - \frac{n^3}{120} \psi_3(n).$$

Theorem 5. *We have*

$$(a) \sum_{m=1}^{n-1} \sigma'_{1,7}(m, n-m) = \frac{176n^9 - 480n}{7680} \psi_{-1}(n) + \frac{n^7}{24} \psi_1(n) - \frac{n}{480} \psi_7(n),$$

$$(b) \sum_{m=1}^{n-1} \sigma'_{3,5}(m, n-m) = \frac{11n^9}{5040} \psi_{-1}(n) - \frac{n^5}{240} \psi_3(n) + \frac{n^3}{504} \psi_5(n).$$

Theorem 6. *We have*

$$(a) \sum_{m=1}^{n-1} \sigma'_{1,11}(m, n-m) = \frac{5223960n^{13} - 20638800n}{495331200} \psi_{-1}(n) + \frac{n^{11}}{24} \psi_1(n)$$

$$- \frac{691n}{65520} \psi_{11}.$$

$$(b) \sum_{m=1}^{n-1} \sigma'_{3,9}(m, n-m) = \frac{n^{13}}{2640} \psi_{-1}(n) - \frac{n^9}{240} \psi_3(n) + \frac{n^3}{264} \psi_9(n).$$

$$(c) \sum_{m=1}^{n-1} \sigma'_{5,7}(m, n-m) = \frac{n^{13}}{10080} \psi_{-1}(n) + \frac{n^7}{504} \psi_5(n) - \frac{n^5}{480} \psi_7(n).$$

It worth at this point to notice the pattern in our formulas, namely:

$$(6) \quad \sum_{m=1}^{n-1} \sigma'_{r,s}(m, n-m) = (An^{r+s+1} + Bn) \psi_{-1}(n) + Cn^r \psi_s(n) + Dn^s \psi_r(n).$$

This motivates the following question.

Problem. Is it true that for all $r, s \in \mathbb{N}$ there exist $A, B, C, D \in \mathbb{Q}$ such that identity (6) holds?

2. PROOF OF MAIN THEOREM

As it was mentioned earlier, Main Theorem is an analogue of Theorem 13.1 in Huard *et al.* [5] and the proof is essentially the same. See also Williams [12, p. 137-140]. To simplify let

$$g(a, b, x, y) = f(a, b, x, y) - f(x, y, a, b)$$

so that

$$(7) \quad g(a, -b, x, y) = g(-a, b, x, y) \quad \text{and} \quad g(a, b, x, y) = -g(x, y, a, b).$$

Combining (7) with the fact that

$$(a, b, x, y) \in \mathcal{B}'(n) \text{ if and only if } (x, y, a, b) \in \mathcal{B}'(n) \text{ if and only if } (y, x, b, a) \in \mathcal{B}'(n),$$

we have

$$\begin{aligned} & \sum_{(a,b,x,y) \in \mathcal{B}'(n)} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y) \\ & \quad - f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y)). \\ = & \sum_{(a,b,x,y) \in \mathcal{B}'(n)} (f(a, b, x, -y) - f(x, -y, a, b) + f(a, a-b, x+y, y) \\ & \quad - f(x, x+y, a-b, a) + f(a-b, a, y, x+y) - f(x+y, y, a, a-b)). \\ = & \sum_{(a,b,x,y) \in \mathcal{B}'(n)} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y) + g(a, b, x, -y)). \end{aligned}$$

On the other hand we have

$$\begin{aligned} & \sum_{\substack{1 \leq t \leq n \\ (t,n)=1}} (f(1, 0, n, t) - f(n, t, 1, 0) + f(0, 1, t, n) \\ & \quad - f(t, n, 0, 1) + f(1, 1, n-t, -t) - f(n-t, -t, 1, 1)). \\ = & \sum_{\substack{1 \leq t \leq n \\ (t,n)=1}} (g(1, 0, n, t) + g(0, 1, t, n) + g(1, 1, n-t, -t)). \end{aligned}$$

Therefore identity (1) simplifies to

$$\begin{aligned} (8) \quad & \sum_{(a,b,x,y) \in \mathcal{B}'(n)} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y) + g(a, b, x, -y)) \\ & = \sum_{\substack{1 \leq t \leq n \\ (t,n)=1}} (g(1, 0, n, t) + g(0, 1, t, n) + g(1, 1, n-t, -t)). \end{aligned}$$

We will consider three cases. First note that $(a, b, x, y) \in \mathcal{B}'(n)$ and $a = b$ means that $a = b = 1$. Therefore, considering the terms with $a = b$ the left hand side

of (8) becomes

$$\begin{aligned}
& \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a=b=1}} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y) + g(a, b, x, -y)) \\
&= \sum_{\substack{x+y=n \\ (x,y)=1}} (g(1, 0, x+y, y) + g(0, 1, y, x+y) + g(1, 1, x, -y)) \\
&= \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (g(1, 0, n, t) + g(0, 1, t, n) + g(1, 1, n-t, -t)).
\end{aligned}$$

Secondly, we consider the terms with $a < b$. Noticing that if $a < b$ and $(a, b, x, y) \in \mathcal{B}'(n)$, then $a(x+y) + (b-a)y \in \mathcal{B}'(n)$ and $(a, (b-a), (x+y), y) \in \mathcal{B}'(n)$, the left hand side of identity (8) becomes

$$\begin{aligned}
& \sum_{\substack{(a,b-a,x+y,y) \in \mathcal{B}'(n) \\ a < b}} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y)) \\
&+ \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a < b}} g(a, b, x, -y) \\
&= \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x > y}} (g(a, -b, x, y) + g(-b, a, y, x)) \\
&+ \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x < y}} g(x, y, a, -b) \\
&= \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x > y}} g(a, -b, x, y) + \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x > y}} g(b, -a, y, x) \\
&- \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x < y}} g(a, -b, x, y) \\
&= - \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x > y}} g(x, y, a, -b) + \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x < y}} g(a, -b, x, y) \\
&- \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x < y}} g(a, -b, x, y) \\
&= - \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x > y}} g(x, y, a, -b)
\end{aligned}$$

Finally, we consider the terms with $a > b$. The left hand side of (8) equals

$$\begin{aligned}
& \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y) + g(a, b, x, -y)) \\
&= \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y)) \\
&+ \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} g(a, b, x, -y).
\end{aligned}$$

But with the help of (7) we find

$$\begin{aligned}
& \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y)) \\
&= \sum_{(a,b,x,x+y) \in \mathcal{B}'(n)} (g(a+b, a, x+y, y) + g(a, a+b, y, x+y)) \\
&= \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ y > x}} (g(a+b, a, y, y-x) + g(a, a+b, y-x, y)) \\
&= \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} (g(x+y, y, a, a-b) + g(y, x+y, a-b, a)) \\
&= - \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y)).
\end{aligned}$$

That is,

$$\sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y)) = 0$$

and thus

$$\begin{aligned}
& \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} (g(a, a-b, x+y, y) + g(a-b, a, y, x+y)) \\
&+ \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} g(a, b, x, -y) \\
&= \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ a > b}} g(a, b, x, -y) = \sum_{\substack{(a,b,x,y) \in \mathcal{B}'(n) \\ x > y}} g(x, y, a, -b)
\end{aligned}$$

which complete the proof.

3. PROOF OF OTHER THEOREMS

Proof of Theorem 2. Let $f(a, b, x, y) = x^2$. Then clearly for all $a, b, x, y \in \mathbb{Z}$

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b).$$

With this choice the left hand side and the right hand side of equation (1) are respectively

$$4 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} xy \quad \text{and} \quad \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (2t^2 - 2nt + 2n^2 - 2).$$

Then by Main Theorem and relation (2) we obtain

$$\begin{aligned} (9) \quad \sum_{m=1}^{n-1} \sigma'_{1,1}(m, n-m) &= \sum_{(a,b,x,y) \in \mathcal{B}'(n)} xy \\ &= \frac{1}{2} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^2 - \frac{n}{2} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t + \frac{n^2-1}{2} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} 1. \end{aligned}$$

Further by the relations in (5) and (3) we find

$$\begin{aligned} (10) \quad \sum_{\substack{1 \leq t < n \\ (t,n)=1}} 1 &= n\psi_{-1}(n) \\ \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t &= \frac{n^2}{2}\psi_{-1}(n) \\ \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^2 &= \frac{n^3}{3}\psi_{-1}(n) + \frac{n}{6}\psi_1(n). \end{aligned}$$

Now by virtue of formulas (9) and (10) we have

$$\begin{aligned} \sum_{m=1}^{n-1} \sigma'_{1,1}(m, n-m) &= \frac{1}{2} \left(\frac{n^3}{3}\psi_{-1}(n) + \frac{n}{6}\psi_1(n) \right) - \frac{n}{2} \left(\frac{n^2}{2}\psi_{-1}(n) \right) + \frac{(n^2-1)n}{2}\psi_{-1}(n) \\ &= \frac{5n^3-6n}{12}\psi_{-1}(n) + \frac{n}{12}\psi_1(n). \end{aligned}$$

This completes the proof.

Proof of Theorem 3. Considering the function $f(a, b, x, y) = x^2y^2$, the left hand side and the right hand side of equation (1) are respectively

$$8 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^3y \quad \text{and} \quad \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (t^4 - 2nt^3 + 3n^2t^2 - 1).$$

Moreover it is easily checked that this function satisfies the condition of Main Theorem and therefore with the help relation (2) we obtain

$$\begin{aligned} (11) \quad \sum_{m=1}^{n-1} \sigma'_{1,3}(m, n-m) &= \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^3y \\ &= \frac{1}{8} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^4 - \frac{n}{4} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^3 + \frac{3n^2}{8} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^2 - \frac{1}{8} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} 1. \end{aligned}$$

Further by the formulas in (5) and (3) we find

$$(12) \quad \begin{aligned} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^3 &= \frac{n^4}{4} \psi_{-1}(n) + \frac{n^2}{4} \psi_1(n) \\ \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^4 &= \frac{n^5}{5} \psi_{-1}(n) + \frac{n^3}{3} \psi_1(n) - \frac{n}{30} \psi_3(n). \end{aligned}$$

Combining formulas (10), (11), and (12) we deduce the desired identity.

Proof of Theorem 4. (a) Application of Main Theorem to the function $f(a, b, x, y) = xy^5 - 10x^3y^3$ gives

$$-108 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} xy^5 = \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (-9t^6 + 30nt^5 - 30n^2t^4 - 10n^3t^3 + n^5t + 9),$$

which by relation (2) yields

$$(13) \quad \begin{aligned} \sum_{m=1}^{n-1} \sigma'_{1,5}(m, n-m) &= \sum_{(a,b,x,y) \in \mathcal{B}'(n)} xy^5 \\ &= \frac{9}{108} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^6 - \frac{30n}{108} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^5 + \frac{30n^2}{108} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^4 \\ &\quad + \frac{10n^3}{108} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^3 - \frac{n^5}{108} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t - \frac{9}{108} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} 1. \end{aligned}$$

Further by (5) and (3) we get

$$(14) \quad \begin{aligned} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^5 &= \frac{n^6}{6} \psi_{-1}(n) + \frac{5n^4}{12} \psi_1(n) - \frac{n^2}{12} \psi_3(n) \\ \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^6 &= \frac{n^7}{7} \psi_{-1}(n) + \frac{n^5}{2} \psi_1(n) - \frac{n^3}{6} \psi_3(n) + \frac{n}{42} \psi_5(n). \end{aligned}$$

Combining formulas (10), (12), (13), and (14) we deduce the result of part (a).

(b) By Main Theorem applied to the function $f(a, b, x, y) = xy^5 - x^3y^3$ we find

$$18 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^3y^3 = \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (3nt^5 - 3n^2t^4 - n^3t^3 + n^5t).$$

Then by the relation (2) we have

$$(15) \quad \begin{aligned} \sum_{m=1}^{n-1} \sigma'_{3,3}(m, n-m) &= \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^3y^3 \\ &= \frac{n}{6} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^5 - \frac{n^2}{6} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^4 - \frac{n^3}{18} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^3 + \frac{n^5}{18} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t, \end{aligned}$$

which by (10), (12), and (14) implies the result.

Proof of Theorem 5. (a) Application of Main Theorem to the function $f(a, b, x, y) = -22x^7y + 112x^5y^3$ gives

$$\begin{aligned} & 1440 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} xy^7 \\ &= \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (90t^8 - 428nt^7 + 658n^2t^6 - 238n^3t^5 - 210n^4t^4 + 462n^5t^3 - 154n^6t^2 - 90), \end{aligned}$$

which by (2) yields

$$\begin{aligned} \sum_{m=1}^{n-1} \sigma'_{1,7}(m, n-m) &= \sum_{(a,b,x,y) \in \mathcal{B}'(n)} xy^7 \\ &= \frac{90}{1440} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^8 - \frac{428n}{1440} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^7 + \frac{658n^2}{1440} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^6 \\ &\quad - \frac{238n^3}{1440} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^5 - \frac{210n^4}{1440} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^4 + \frac{462n^5}{1440} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^3 \\ &\quad - \frac{154n^6}{1440} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^2 - \frac{90}{1440} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} 1. \end{aligned} \tag{16}$$

Further by the relations in (5) and (3) we find

$$\begin{aligned} \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^7 &= \frac{n^8}{8} \psi_{-1}(n) + \frac{7n^6}{12} \psi_1(n) - \frac{7n^4}{24} \psi_3(n) + \frac{7n^2}{84} \psi_5(n) \\ \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^8 &= \frac{n^9}{9} \psi_{-1}(n) + \frac{2n^7}{3} \psi_1(n) - \frac{7n^5}{15} \psi_3(n) + \frac{2n^3}{9} \psi_5(n) - \frac{n}{30} \psi_7(n). \end{aligned} \tag{17}$$

Now use (10), (12), (14), (16), and (17) to conclude the result.

(b) By Main Theorem applied to the function $f(a, b, x, y) = x^7y - x^5y^3$ we have

$$90 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^3y^5 = \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (-nt^7 + 11n^2t^6 - 26n^3t^5 + 30n^4t^4 - 21n^5t^3 + 7n^6t^2).$$

This fact combined with the relations (10), (12), (14), and (17) gives the result.

Proof of Theorem 6. (a) Considering the function $f(a, b, x, y) = 271x^{11}y - 1540x^9y^3 + 1584x^7y^5$ in Main Theorem we have

$$\begin{aligned} 7560 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} xy^{11} &= \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (315t^{12} + 62nt^{11} - 7271n^2t^{10} + 27665n^3t^9 \\ &\quad - 49170n^4t^8 + 37158n^5t^7 + 6930n^6t^6 - 33990n^7t^5 \\ &\quad + 30855n^8t^4 - 14905n^9t^3 + 2981n^{10}t^2 - 315). \end{aligned}$$

Combining this fact with the identities (10), (12), (14), (17) along with

$$\begin{aligned}
\sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^9 &= \frac{n^{10}}{10} \psi_{-1}(n) + \frac{3n^8}{4} \psi_1(n) - \frac{7n^6}{10} \psi_3(n) + \frac{n^4}{2} \psi_5(n) \\
&\quad - \frac{3n^2}{20} \psi_7(n) \\
\sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^{10} &= \frac{n^{11}}{11} \psi_{-1}(n) + \frac{5n^9}{6} \psi_1(n) - n^7 \psi_3(n) + n^5 \psi_5(n) \\
&\quad - \frac{n^3}{2} \psi_7(n) + \frac{5n}{66} \psi_9(n) \\
(18) \quad \sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^{11} &= \frac{n^{12}}{12} \psi_{-1}(n) + \frac{11n^{10}}{12} \psi_1(n) - \frac{11n^8}{8} \psi_3(n) + \frac{11n^6}{6} \psi_5(n) \\
&\quad - \frac{11n^4}{8} \psi_7(n) + \frac{5n^2}{12} \psi_9(n) \\
\sum_{\substack{1 \leq t < n \\ (t,n)=1}} t^{12} &= \frac{n^{13}}{13} \psi_{-1}(n) + n^{11} \psi_1(n) - \frac{11n^9}{6} \psi_3(n) + \frac{22n^7}{7} \psi_5(n) \\
&\quad - \frac{33n^5}{10} \psi_7(n) + \frac{5n^3}{3} \psi_9(n) - \frac{691n}{2730} \psi_{11}(n)
\end{aligned}$$

yields the result.

(b) As to this part, using the function $f(a, b, x, y) = -2x^{11}y + 11x^9y^3 - 9x^7y^5$ in Main Theorem, we find

$$\begin{aligned}
180 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^3y^9 &= \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (-16nt^{11} + 97n^2t^{10} - 268n^3t^9 + 411n^4t^8 - 282n^5t^7 \\
&\quad - 63n^6t^6 + 264n^7t^5 - 231n^8t^4 + 110n^9t^3 - 22n^{10}t^2).
\end{aligned}$$

Now use the relations (2), (10), (12), (14), (17), and (18) to deduce the desired formula.

(c) Use the function $f(a, b, x, y) = 8x^{11}y - 35x^9y^3 + 27x^7y^5$ in Main Theorem to obtain

$$\begin{aligned}
2520 \sum_{(a,b,x,y) \in \mathcal{B}'(n)} x^5y^7 &= \sum_{\substack{1 \leq t < n \\ (t,n)=1}} (46nt^{11} - 253n^2t^{10} + 640n^3t^9 - 825n^4t^8 + 174n^5t^7 \\
&\quad + 945n^6t^6 - 1380n^7t^5 + 1005n^8t^4 - 440n^9t^3 + 88n^{10}t^2)
\end{aligned}$$

and argue as before to conclude the result.

4. CONCLUDING REMARKS ON INTEGER REPRESENTATIONS

In this section we will show that each of the sums $\sum_{m=1}^{n-1} \sigma_r(m) \sigma_s(n-m)$ and $\sum_{m=1}^{n-1} \sigma'_{r,s}(m, n-m)$ count certain representations of positive integers involving radicals.

Definition 2. For $1 < n \in \mathbb{N}$ and $r, s \in \mathbb{N}$ let the functions $L_{r,s}(n)$, $M_{r,s}(n)$, $L'_{r,s}(n)$ and $M'_{r,s}(n)$ be defined as follows

$$\begin{aligned} L_{r,s}(n) &= \#\{(a, b, c, d, x, y) \in \mathbb{N}_0^2 \times \mathbb{N}^4 : (\sqrt[r]{a+c}, \sqrt[s]{b+d}, x, y) \in \mathcal{B}(n)\} \\ M_{r,s}(n) &= \#\{(a, b, c, d, x, y, k, l) \in \mathbb{N}_0^2 \times \mathbb{N}^6 : (\sqrt[r]{k(a+c)}, \sqrt[s]{l(b+d)}, x, y) \in \mathcal{B}(n) \\ &\quad \text{and } (a, c) = (b, d) = 1\}. \\ L'_{r,s}(n) &= \#\{(a, b, c, d, x, y) \in \mathbb{N}_0^2 \times \mathbb{N}^4 : (\sqrt[r]{a+c}, \sqrt[s]{b+d}, x, y) \in \mathcal{B}'(n)\} \\ M'_{r,s}(n) &= \#\{(a, b, c, d, x, y, k, l) \in \mathbb{N}_0^2 \times \mathbb{N}^6 : (\sqrt[r]{k(a+c)}, \sqrt[s]{l(b+d)}, x, y) \in \mathcal{B}'(n) \\ &\quad \text{and } (a, c) = (b, d) = 1\}. \end{aligned}$$

Theorem 7. If $1 < n \in \mathbb{N}$ and $r, s \in \mathbb{N}$, then

$$\begin{aligned} (a) \quad L_{r,s}(n) &= M_{r,s}(n) = \sum_{m=1}^{n-1} \sigma_r(m) \sigma_s(n-m). \\ (b) \quad L'_{r,s}(n) &= M'_{r,s}(n) = \sum_{m=1}^{n-1} \sigma_{r,s}(m, n-m). \end{aligned}$$

Proof. (a) First note that

$$(19) \quad \sum_{m=1}^{n-1} \sigma_r(m) \sigma_s(n-m) = \sum_{(a,b,x,y) \in \mathcal{B}'(n)} a^r b^s.$$

Further we have

$$\begin{aligned} L_{r,s}(n) &= \sum_{\substack{(a,b,c,d,x,y) \in \mathbb{N}_0^2 \times \mathbb{N}^4 \\ (\sqrt[r]{a+c}, \sqrt[s]{b+d}, x, y) \in \mathcal{B}(n)}} 1 \\ &= \sum_{(u,v,x,y) \in \mathcal{B}(n)} \left(\sum_{\substack{(a,c) \in \mathbb{N}_0 \times \mathbb{N} \\ a+c=u^r}} 1 \right) \left(\sum_{\substack{(b,d) \in \mathbb{N}_0 \times \mathbb{N} \\ b+d=v^s}} 1 \right) 1 \\ &= \sum_{(u,v,x,y) \in \mathcal{B}(n)} u^r v^s \end{aligned}$$

and

$$\begin{aligned} M_{r,s}(n) &= \sum_{\substack{(a,b,c,d,x,y,k,l) \in \mathbb{N}_0^2 \times \mathbb{N}^6 \\ (\sqrt[r]{k(a+c)}, \sqrt[s]{l(b+d)}, x, y) \in \mathcal{B}(n) \\ (a,c)=(b,d)=1}} 1 \\ &= \sum_{(u,v,x,y) \in \mathcal{B}(n)} \left(\sum_{\substack{e|u^r \\ (a,c) \in \mathbb{N}_0 \times \mathbb{N} \\ a+c=e \\ (a,c)=1}} 1 \right) \left(\sum_{\substack{f|v^s \\ (b,d) \in \mathbb{N}_0 \times \mathbb{N} \\ b+d=f \\ (b,d)=1}} 1 \right) \\ &= \sum_{(u,v,x,y) \in \mathcal{B}(n)} \left(\sum_{e|u^r} \phi(e) \right) \left(\sum_{f|v^s} \phi(f) \right) \\ &= \sum_{(u,v,x,y) \in \mathcal{B}(n)} u^r v^s, \end{aligned}$$

which with the help of (19) completes the proof of part (a).

(b) This part follows similarly with an application of relation (2) instead of relation (19). \square

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